# Elementary Row Operations and Linear Equations

## Linear Algebra

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# Overview

- Matrix Multiplication
- Elementary Row Operations
- Elementary Matrices
- Linear Equations

# Matrix Multiplication

# Basic Notation

- By $A \in \mathbb{R}^{m \times n}$ we denote a matrix with m rows and n columns, where the entries of A are real numbers.

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} - & a_1^T & - \\ - & a_2^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix}$$

# Matrix–Vector Multiplication

- If we write A by rows, then we can express Ax as,

$$A \in \mathbb{R}^{m\times n} \qquad y = Ax = \begin{bmatrix} - & a_1^T & - \\ - & a_2^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix} x = \begin{bmatrix} a_1^T x \\ a_2^T x \\ \vdots \\ a_m^T x \end{bmatrix}. \qquad a_i^T x = \sum_{j=1}^{n} a_{ij} x$$

- If we write A by columns, then we have:

$$y = Ax = \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = [a_1]x_1 + [a_2]x_2 + \cdots + [a_n]x_n .$$

  - y is a **linear combination** of the columns A.

**columns of A are linearly independent if Ax = 0 implies x = 0**

It is also possible to multiply on the left by a row vector.

- If we write A by columns, then we can express $x^T A$ as,

$$y^T = x^T A = x^T \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} x^T a_1 & x^T a_2 & \cdots & x^T a_n \end{bmatrix}$$

- expressing A in terms of rows we have:

$$y^T = x^T A = \begin{bmatrix} x_1 & x_2 & \cdots & x_m \end{bmatrix} \begin{bmatrix} - & a_1^T & - \\ - & a_2^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix}$$

$$= x_1 \begin{bmatrix} - & a_1^T & - \end{bmatrix} + x_2 \begin{bmatrix} - & a_2^T & - \end{bmatrix} + \cdots + x_m \begin{bmatrix} - & a_m^T & - \end{bmatrix}$$

- $y^T$ is a linear combination of the rows of A.

# Matrix–Vector Multiplication

- Properties
  - $A(u + v) = Au + Av$
  - $(A + B)u = Au + Bu$
  - $(\alpha A)u = \alpha(Au) = A(\alpha u) = \alpha Au$
  - $0u = 0$
  - $A0 = 0$
  - $Iu = u$

# Matrix–Vector Multiplication

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} = \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix} = \begin{bmatrix} - & a_1^T & - \\ - & a_2^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix}$$

## Example: Write in matrix–vector multiplication

- Column $j$: $a_j =$
- Row $i$: $a_i^T =$
- Vector sum of rows of $A =$
- Vector sum of columns of $A =$

$$A = \begin{bmatrix} -1 & 2 & 1 \\ 2 & 0 & -2 \end{bmatrix}$$

# Matrix–Matrix Multiplication

## Definition

Let $A$ be an $m \times n$ matrix over the field $F$ and let $B$ be an $n \times p$ matrix over $F$. The product $AB$ is the $m \times p$ matrix $C$ whose $i, j$ entry is:

$$C_{ij} = \sum_{r=1}^{n} A_{ir} B_{rj}$$

# Matrix–Matrix Multiplication

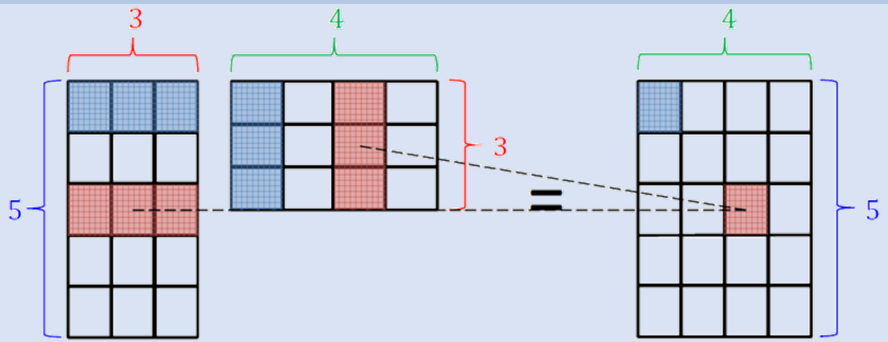
- $A \in \mathbb{R}^{m \times n}, B \in \mathbb{R}^{n \times p} \rightarrow \mathbb{R}^{m \times p}$
  - $a_i$ rows of A, $b_j$ cols of B

$$C = AB \qquad for\ 1 \le i \le m\ , 1 \le j \le p$$

$$inner\ product\ (a_i . b_j)$$

$$C_{ij} = a_i^T b_j$$

## Example



# Matrix–Matrix Multiplication (different views)

1. As a set of vector–vector products

$$C = AB = \begin{bmatrix} - & a_1^T & - \\ - & a_2^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix} \begin{bmatrix} | & | & & | \\ b_1 & b_2 & \cdots & b_p \\ | & | & & | \end{bmatrix} = \begin{bmatrix} a_1^T b_1 & a_1^T b_2 & \cdots & a_1^T b_p \\ a_2^T b_1 & a_2^T b_2 & \cdots & a_2^T b_p \\ \vdots & \vdots & \ddots & \vdots \\ a_m^T b_1 & a_m^T b_2 & \cdots & a_m^T b_p \end{bmatrix}$$

2. As a sum of outer products

$$C = AB = \begin{bmatrix} | & | & & | \\ a_1 & a_2 & \cdots & a_n \\ | & | & & | \end{bmatrix} \begin{bmatrix} - & b_1^T & - \\ - & b_2^T & - \\ & \vdots & \\ - & b_n^T & - \end{bmatrix} = \sum_{i=1}^{n} a_i b_i^T$$

# Matrix–Matrix Multiplication (different views)

3. As a set of matrix–vector products.

$$C = AB = A\begin{bmatrix} | & | & & | \\ b_1 & b_2 & \cdots & b_p \\ | & | & & | \end{bmatrix} = \begin{bmatrix} | & | & & | \\ Ab_1 & Ab_2 & \cdots & Ab_p \\ | & | & & | \end{bmatrix}$$

Here the *i*th column of C is given by the matrix–vector product with the vector on the right, $c_i = Ab_i$. These matrix–vector products can in turn be interpreted using both viewpoints given in the previous subsection.

4. As a set of vector–matrix products.

$$C = AB = \begin{bmatrix} - & a_1^T & - \\ - & a_2^T & - \\ & \vdots & \\ - & a_m^T & - \end{bmatrix} B = \begin{bmatrix} - & a_1^T B & - \\ - & a_2^T B & - \\ & \vdots & \\ - & a_m^T B & - \end{bmatrix}$$

# Matrix–Matrix Multiplication

- Properties:
  - Associative

$$(AB)C = A(BC)$$

  - Distributive

$$A(B + C) = AB + AC$$

  - NOT commutative

$$AB \neq BA$$

    – Dimensions may not even be conformable

## Theorem

If $A, B, C$ are matrices over the field $F$ such that the products $BC$ and $A(BC)$ is defined, then so are the products $AB$, $(AB)C$ and

$$A(BC) = (AB)C$$

**Proof:**

## Note

Linear combinations of linear combinations of the rows of C are again linear combinations of the rows of C

# Matrix Power

- $A^k$ : repeated multiplication of a square matrix

$$A^1 = A, A^2 = AA, \dots, A^k = \underbrace{AA \cdots A}_{k \text{ matrices}}$$

- Properties:
  - $A^j A^k = A^{j+k}$
  - $(A^j)^k = A^{jk}$

where j and k are non-negative integers and $A^0$ is assumed to be I

- For diagonal matrices:

$$D = \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix} \Rightarrow D^k = \begin{bmatrix} d_1^k & 0 & \cdots & 0 \\ 0 & d_2^k & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & d_n^k \end{bmatrix}$$

# Elementary Row Operations

- Elementary Row Operations
  1. Scaling: Multiply all entries in a row by a nonzero scalar.
  2. Replacement: Replace one row by the sum of itself and a multiple of another row.
  3. Interchange: Interchange two rows.

- Elementary Row Operation is a special type of function $e$ on $m \times n$ matrix $A$ and gives an $m \times n$ matrix $e(A)$
  1. Scaling : $e(A)_{\mathrm{ij}} = \mathrm{cA}_{\mathrm{ij}}$
  2. Replacement: $e(A)_{\mathrm{ij}} = A_{ij} + \mathrm{cA}_{\mathrm{kj}}$
  3. Interchange: $e(A)_{\mathrm{ij}} = \mathrm{A}_{\mathrm{kj}}$ , $e(A)_{\mathrm{kj}} = \mathrm{A}_{\mathrm{ij}}$

**In defining $e(A)$, it is not really important how many columns A has, but the number of rows of $A$ is crucial.**

# Inverse of Elementary Row Operation

**Theorem**

The inverse operation (function) of an elementary row operation exists and is an elementary row operation of the same type.

**Proof:**

# Row-Equivalent

## Definition

If $A$ and $B$ are $m \times n$ matrices over the field $F$, we say that $B$ is **row-equivalent** to $A$ if $B$ can be obtained from $A$ by a finite sequence of elementary row operations.

## Note (from pervious theorem and this definition)

- Each matrix is row-equivalent to itself
- If $B$ is row-equivalent to A, then A is row-equivalent to B.
- If $B$ is row-equivalent to A, C is row-equivalent to B, then $C$ is row-equivalent to $A$

# Elementary Matrices

# Elementary Matrices

## Definition

A $m \times n$ matrix is an elementary matrix if it can be obtained from the $m \times m$ identity matrix by means of a single elementary row operation.

## Example

Find all $2 \times 2$ elementary matrices.

### Theorem

Let $e$ be an elementary row operation and let $E$ be the $m \times m$ elementary matrix $E = e(I)$. Then, for every $m \times n$ matrix A:

$$e(A) = EA$$

**Proof:**

**Multiplication of a matrix on the left by a square matrix performs row operations.**

# Elementary Matrices

## Example

(From theorem)

$$M_4(M_3(M_2(M_1A))) = \left(M_4\left(M_3(M_2M_1)\right)\right)A$$

| Matrix | Elementary row operation | Elementary matrix |
| --- | --- | --- |
| $\begin{bmatrix} 1 & 0 & 2 \\ -2 & 0 & -3 \\ 0 & 2 & 0 \end{bmatrix}$ | $R_2 \leftarrow R_2 + 2R_1$ | $M_1 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ |
| $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 2 & 0 \end{bmatrix}$ | $R_2 \leftrightarrow R_3$ | $M_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ |
| $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ | $R_2 \leftarrow \frac{1}{2}R_2$ | $M_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$ |
| $\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ | $R_1 \leftarrow R_1 + (-2)R_3$ | $M_4 = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ |
| $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ | | |

# Row–Equivalent and Elementary Matrices

**Theorem**

Let A and B be $m \times n$ matrices. Then $B$ is row–equivalent to $A$ if and only if $B = PA$, where $P$ is a product of $m \times m$ elementary matrices.

# Linear Equations

# Systems of Linear Equations

## Definition

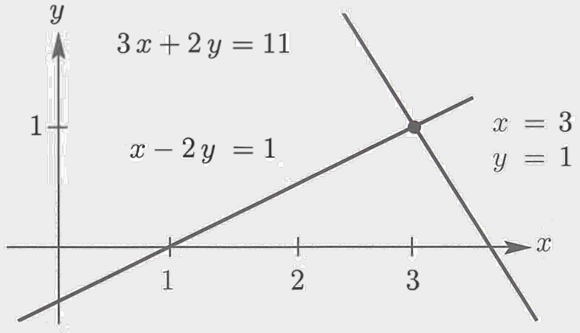
A system of m linear equations with n unknowns:

- $F$ is a field, we want to find n scalars (elements of $F$) $x_1, \dots, x_n$ which satisfy the conditions: ($A_{ij}, y_k$ are elements of $F$)

$$A_{11}x_1 + A_{12}x_2 + \cdots + A_{1n}x_n = y_1$$
$$A_{21}x_1 + A_{22}x_2 + \cdots + A_{2n}x_n = y_2$$
$$\dots$$
$$A_{m1}x_1 + A_{m2}x_2 + \cdots + A_{mn}x_n = y_m$$

If $y_1 = y_2 = \cdots = y_m = 0$, we say that the system is homogeneous.

A solution of this system of linear equations is vector $\begin{bmatrix} s_1 \\ \vdots \\ s_n \end{bmatrix}$ whose components satisfy $x_1 = s_1, \dots, x_n = s_n$




- Consider this simple system of equations,

$$x - 2y = 1$$
$$3x + 2y = 11$$

- Can be expressed as a matrix–vector multiplication
- Matrix Equation: $Ax=b$

$$\underbrace{\begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 1 \\ 11 \end{bmatrix}}_{b}$$

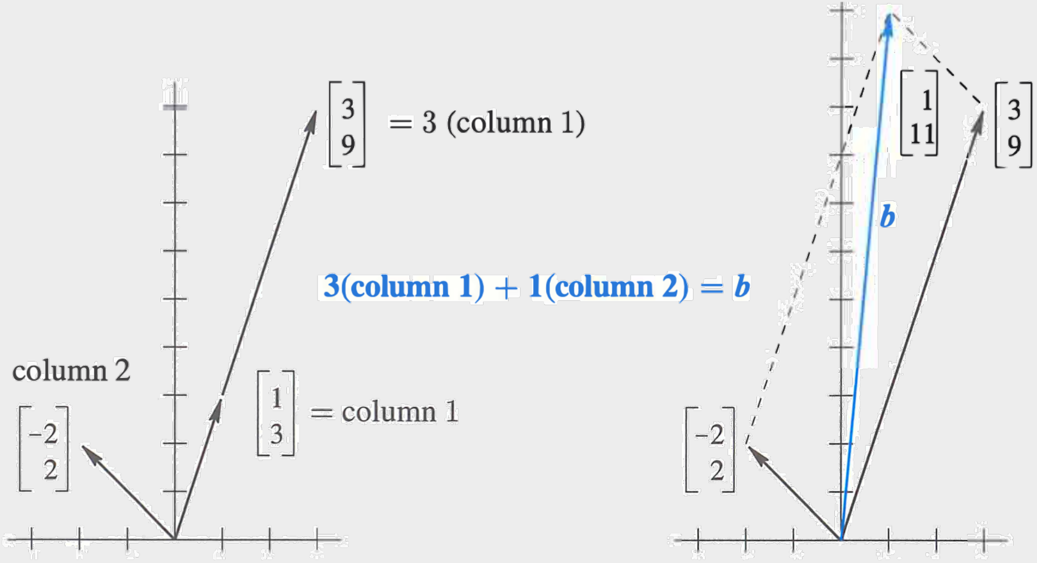
- $A$ is often called coefficient matrix: $\begin{bmatrix} 1 & -2 \\ 3 & 2 \end{bmatrix}$
- $Ab$ is an Augmented matrix: $\begin{bmatrix} 1 & -2 & 1 \\ 3 & 2 & 11 \end{bmatrix}$

- Also, Can be expressed as linear combination of cols:

$$x - 2y = 1$$
$$3x + 2y = 11$$

$$\underbrace{\begin{bmatrix}1 & -2\\ 3 & 2\end{bmatrix}}_{A}\underbrace{\begin{bmatrix}x\\ y\end{bmatrix}}_{x} = \underbrace{\begin{bmatrix}1\\ 11\end{bmatrix}}_{b}$$

$$x\begin{bmatrix}1\\ 3\end{bmatrix} + y\begin{bmatrix}-2\\ 2\end{bmatrix} = \begin{bmatrix}1\\ 11\end{bmatrix} = b$$




- Same for $n$ equation, $n$ variable

- Subtract a multiple of equation (1) from (2) to eliminate a variable

$$\begin{aligned} x - 2y &= 1 \\ 3x + 2y &= 11 \end{aligned}$$

multiply equation 1 by 3

Subtract to eliminate $3x$

$$\begin{aligned} x - 2y &= 1 \\ 8y &= 8 \end{aligned}$$

$$\underbrace{\begin{bmatrix} 1 & -2 \\ 0 & 8 \end{bmatrix}}_{U} \underbrace{\begin{bmatrix} x \\ y \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 1 \\ 8 \end{bmatrix}}_{c}$$

$A$ has become a upper triangle matrix $U$

# Idea Of Elimination (Row Reduction Algorithm)

- The **pivots** are on the diagonal of the triangle after elimination (boldface 2 below is the first pivot)

$$\begin{aligned} 2x + 4y - 2z &= 2 \\ 4x + 9y - 3z &= 8 \\ -2x - 3y + 7z &= 10 \end{aligned} \quad \Rightarrow \quad \begin{aligned} 2x + 4y - 2z &= 2 \\ 1y + 1z &= 4 \\ 4z &= 8 \end{aligned}$$

- Step 1: subtract $(1)$ from $(2)$ to eliminate x's in $(2)$
- Step 2: subtract $(1)$ from $(3)$ to totally eliminate x
- Step 3: subtract new $(2)$ from new $(3)$

**Definition**

The variables corresponding to pivot columns in the matrix are called **basic variables**.
The other variables are called a **free variable**.

$$\begin{bmatrix} x & x & x & x \\ 0 & x & x & x \\ 0 & 0 & x & x \\ 0 & 0 & 0 & x \end{bmatrix} \qquad \begin{bmatrix} x & x & x & x \\ 0 & x & x & x \\ 0 & 0 & 0 & x \\ 0 & 0 & 0 & x \end{bmatrix}$$

# Homogenous system

**Theorem**

If $A$ and $B$ are row-equivalent $m \times n$ matrices, the homogenous systems of linear equations $Ax = 0$ and B$x = 0$ have exactly the same solutions.

**Proof:**

### Example

Find the solution for this system.
Suppose $F$ is the field of complex number and the coefficient matrix is:

$$A = \begin{bmatrix} -1 & i \\ -i & 3 \\ 1 & 2 \end{bmatrix}$$

# Solution of system of linear equations

## Definition

The two systems of linear equations are equivalent if each equation in each system is a linear combination of the equations in other system.

## Theorem

Equivalent systems of linear equations have exactly the same solutions.

**Proof:**

## Note

- It is important to note that row operations are reversible. If two rows are interchanged, they can be returned to their original positions by another interchange.
- If the augmented matrices of two linear systems are row equivalent, then the two systems have the same solution set.

# Existence and Uniqueness Questions

- A system of linear equations has:
  - No solution → inconsistent
  - Exactly one solution
  - Infinitely many solutions } → consistent

Next session:

1. Is the system consistent? That is, does at least one solution exist?
2. If a solution exists, is it the only one? That is, is the solution unique?

# Conclusion

- Different view of matrix multiplication
- Linear combination and matrix multiplication
- Associativity of three matrices multiplication
- Gaussian Elimination
- Row-equivalent of two matrices
- Elementary matrices

- System of linear equations
- Equivalent systems of linear equations have exactly the same solutions.

# References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